

The residue theorem from a numerical perspective

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Abstract

A short vignette illustrating Cauchy's integral theorem using numerical integration

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In this very short vignette, I will use contour integration to evaluate

$$\int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx \quad (1)$$

using numerical methods. This document is part of the **elliptic** package [Hankin \(2006\)](#).

The residue theorem tells us that the integral of $f(z)$ along any closed nonintersecting path is equal to $2\pi i$ times the sum of the residues inside it.

Take a semicircular path P from $-R$ to $+R$ along the real axis, then following a semicircle in the upper half plane, of radius R to close the loop. Now consider large R . Then P encloses a pole at i [there is one at $-i$ also, but this is outside P , so irrelevant here] at which the residue is $-i/2e$. Thus

$$\oint_P f(z) dz = 2\pi i \cdot (-i/2e) = \pi/e \quad (2)$$

along P ; the contribution from the semicircle tends to zero as $R \rightarrow \infty$; thus the integral along the real axis is the whole path integral, or π/e .

We can now reproduce this result analytically. First, choose R :

```
> R <- 400
```

And now define a path P . First, the semicircle:

```
> u1 <- function(x){R*exp(pi*1i*x)}  
> u1dash <- function(x){R*pi*1i*exp(pi*1i*x)}
```

and now the straight part along the real axis:

```
> u2 <- function(x){R*(2*x-1)}  
> u2dash <- function(x){R*2}
```

And define the function:

```
> f <- function(z){exp(1i*z)/(1+z^2)}
```

Now carry out the path integral. I'll do it explicitly, but note that the contribution from the first integral should be small:

```
> answer.approximate <-
+   integrate.contour(f,u1,u1dash) +
+   integrate.contour(f,u2,u2dash)
```

And compare with the analytical value:

```
> answer.exact <- pi/exp(1)
> abs(answer.approximate - answer.exact)
```

```
[1] 6.244969e-07
```

Now try the same thing but integrating over a triangle instead of a semicircle, using `integrate.segments()`. Use a path P' with base from $-R$ to $+R$ along the real axis, closed by two straight segments, one from $+R$ to iR , the other from iR to $-R$:

```
> abs(integrate.segments(f,c(-R,R,1i*R))- answer.exact)
```

```
[1] 5.157772e-07
```

Observe how much better one can do by integrating over a big square instead:

```
> abs(integrate.segments(f,c(-R,R,R+1i*R, -R+1i*R))- answer.exact)
```

```
[1] 2.319341e-08
```

0.1. Residue theorem

Function `residue()` is a wrapper that takes a function $f(z)$ and integrates $f(z)/(z - z_0)$ around a closed loop which encloses z_0 . If $f(\cdot)$ is holomorphic within C , Cauchy's residue theorem states that

$$\oint_C \frac{f(z)}{z - z_0} = f(z_0) \quad (3)$$

and we can test this numerically:

```
> f <- function(z){sin(z)}
> numerical <- residue(f,z0=1,r=1)
> exact <- sin(1)
> abs(numerical-exact)
```

```
[1] 3.91766e-18
```

which is unreasonably accurate, IMO.

References

Hankin RKS (2006). “Introducing elliptic, an R package for elliptic and modular functions.”
Journal of Statistical Software, **15**(7).

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