Introduction to QPA
Part 1

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Third GAP Days

Outline

What is QPA?

- QPA = Quivers and Path Algebras
- GAP package for computations with quotients of path algebras and their modules

Obtaining QPA

- QPA is distributed with GAP (from version 4.7.8)
- Can also clone git repository from

  https://github.com/gap-system/qpa

  to follow QPA development
- Loading QPA in GAP:

```
gap> LoadPackage("qpa");
```

Basic structures

- Quivers
- Path algebras (modulo relations)
- Modules (representations)
  and homomorphisms

Quivers

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & 2 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[loop above, "a"] \arrow[rr, "b"] & & 2 \arrow[dl, "c"] \\ & 3 \arrow[ul, "d"] & \end{tikzcd}$$

Quivers

$$
\begin{tikzcd}
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\end{tikzcd}
\qquad
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\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[loop above, "a"] \arrow[rr, "b"] & & 2 \arrow[dl, "c"] \\
& 3 \arrow[ul, "d"] &
\end{tikzcd}
$$

Quiver: oriented graph (loops and multiple edges allowed)

Paths

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Paths in Q:

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Paths in Q:

- Length 0: e_1, e_2, e_3 (vertices/trivial paths)

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- Length 1: a, b (arrows)

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Paths in Q:

- Length 0: e_1, e_2, e_3 (vertices/trivial paths)
- Length 1: a, b (arrows)
- Length 2: ab (concatenation of a and b)

Paths

$$Q\colon\ 1 \underset{b}{\overset{a}{\rightrightarrows}} 2$$

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Paths in Q:

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- Length 1: a, b, c, d
- Length 2: a^2, ab, bc, cd, da, db

Paths

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- Length 1: a, b, c, d
- Length 2: a^2, ab, bc, cd, da, db
- Length 3: $a^3, a^2b, abc, bcd, cda, cdb, da^2, dab, dbc$

Paths

$$\begin{tikzcd}
Q\colon & 1 \arrow[loop above, "a"] \arrow[rr, "b"] & & 2 \arrow[dl, "c"] \\
& & 3 \arrow[ul, "d"] &
\end{tikzcd}$$

Paths in Q:

- Length 0: e_1, e_2, e_3
- Length 1: a, b, c, d
- Length 2: a^2, ab, bc, cd, da, db
- Length 3: $a^3, a^2b, abc, bcd, cda, cdb, da^2, dab, dbc$
- $\ldots$

Constructing a quiver in QPA

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);
<quiver with 3 vertices and 2 arrows>
```

Path algebras

- Given quiver Q and field k
- Define *path algebra* kQ
- Basis: paths in Q
- Multiplication: concatenation of paths

Path algebras

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

k a field

Path algebras

$$\left.\begin{array}{l} Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \\ k \text{ a field} \end{array}\right\} \rightsquigarrow \text{path algebra } kQ$$

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- Basis: $\{e_1, e_2, e_3, a, b, ab\}$
- Multiplication:

$$\begin{aligned} e_1 \cdot e_1 &= e_1 \\ e_1 \cdot e_2 &= 0 \\ e_1 \cdot a &= a \\ e_2 \cdot a &= 0 \\ a \cdot b &= ab \end{aligned}$$

Path algebras

$$\left.\begin{array}{l} Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \\ k \text{ a field} \end{array}\right\} \rightsquigarrow \text{path algebra } kQ$$

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- Multiplication:

$$e_1 \cdot e_1 = e_1$$
$$e_1 \cdot e_2 = 0$$
$$e_1 \cdot a = a$$
$$e_2 \cdot a = 0$$
$$a \cdot b = ab$$

$\cdot$	e_1	e_2	e_3	a	b	ab
e_1	e_1	0	0	a	0	ab
e_2	0	e_2	0	0	b	0
e_3	0	0	e_3	0	0	0
a	0	a	0	0	ab	0
b	0	0	b	0	0	0
ab	0	0	ab	0	0	0

Constructing a path algebra in QPA

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gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);
<quiver with 3 vertices and 2 arrows>
gap> kQ := PathAlgebra(Rationals, Q);
<Rationals[<quiver with 3 vertices and 2 arrows>]>
```

Relations

- *Relation*: linear combination of paths with common source and target

$$Q: \begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "c"] & 2 \arrow[d, "b"] \\ 3 \arrow[r, "d"] & 4 \end{tikzcd} \qquad \sigma = \underbrace{ab - 2cd}_{\substack{\text{source: } 1 \\ \text{target: } 4}} \in kQ$$

Quotient of path algebra modulo relations

- Given path algebra kQ and a set $\rho \subseteq kQ$ of relations.
- Can create quotient algebra $kQ/\langle \rho \rangle$.

$$Q\colon \begin{array}{ccc} 1 & \xrightarrow{a} & 2 \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle b} \\ 3 & \xrightarrow{d} & 4 \end{array} \qquad A = kQ/\langle ab - 2cd \rangle$$

Quotients in QPA

$$Q\colon\ \begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "c"] & 2 \arrow[d, "b"] \\ 3 \arrow[r, "d"] & 4 \end{tikzcd} \qquad A = kQ/\langle ab - 2cd \rangle$$

```
gap> Q := Quiver(4,[[1,2,"a"],[2,4,"b"],
                    [1,3,"c"],[3,4,"d"]]);;
gap> kQ := PathAlgebra(Rationals,Q);;
gap> A := kQ/[kQ.a*kQ.b - 2*kQ.c*kQ.d];;
gap> A.a * A.b;
[(1)*a*b]
gap> A.c * A.d;
[(1/2)*a*b]
```

Admissible ideals

- $J \subseteq kQ$ ideal generated by the arrows
- Ideal $I \subseteq kQ$ *admissible* if $J^t \subseteq I \subseteq J^2$ for some $t \geq 2$
- If I admissible, then kQ/I finite-dimensional

Admissible ideals – what does it mean?

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long paths must die

$$Q: \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd} \qquad \rho = \{abcab\}$$

Admissible ideals – what does it mean?

$$J^t \subseteq I \subseteq J^2$$

long paths must die

no single arrows in relations

$$Q\colon \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] & & 3 \arrow[ll, "c"] \end{tikzcd} \qquad \rho = \{abcab\}$$

$$Q\colon \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[rr, "c"] & & 3 \end{tikzcd} \qquad \rho = \{ab - c\}$$

Admissible ideals – why?

Algebras of the form kQ/I with I admissible . . .

- . . . are “almost all” finite-dimensional algebras, and
- . . . have a nice theory.

Checking ideals for admissibility in QPA

```
gap> Q := Quiver(4,[[1,2,"a"],[2,4,"b"],
                    [1,3,"c"],[3,4,"d"]]);;
gap> kQ := PathAlgebra(Rationals,Q);;
gap> I := Ideal(kQ, [kQ.a*kQ.b - 2*kQ.c*kQ.d]);;
gap> IsAdmissibleIdeal(I);
true
```

Algebras: summary

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$$Q:\quad \begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "c"'] & 2 \arrow[d, "b"] \\ 3 \arrow[r, "d"'] & 4 \arrow[ul, "x"'] \end{tikzcd}$$

Algebras: summary

Take ...

- a QUIVER Q,
- a FIELD k,

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Algebras: summary

Take ...

- a QUIVER Q,
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- a set $\rho \subseteq kQ$ of RELATIONS

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$$\rho = \{ab - cd, dxa\}$$

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Take ...

- a QUIVER Q,
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$$J^t \subseteq \langle \rho \rangle \subseteq J^2$$

Algebras: summary

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$$\rho = \{ab - cd, dxa\}$$

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Let ... $$A = kQ/\langle \rho \rangle$$

Modules and representations

$$\operatorname{mod} kQ \simeq \operatorname{Rep}_k Q$$

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finitely generated kQ-modules

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representations of Q over k

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Representations

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Want to make a representation R of Q.

$$R\colon\ V_1 \longrightarrow V_2 \longrightarrow V_3$$

Start with the quiver, and put

- a vector space at each vertex,

Representations

Given

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

Want to make a representation R of Q.

$$R\colon\ V_1 \xrightarrow{f_a} V_2 \xrightarrow{f_b} V_3$$

Start with the quiver, and put

- a vector space at each vertex,
- a linear transformation on each arrow.

A representation

$$Q\colon\ 1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

$$R\colon\ k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

Creating a representation in QPA

$$R\colon\; k \xrightarrow{(2\ 0)} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"],[2,3,"b"]]);;
gap> kQ := PathAlgebra(Rationals, Q);;
gap> M := RightModuleOverPathAlgebra
          (kQ, [1,2,1],
           [["a", [[2,0]]], ["b", [[4],[-1]]]]);
<[ 1, 2, 1 ]>
```

Module structure of a representation

$$R\colon\ k \xrightarrow{\begin{pmatrix} 2 & 0 \end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix} 4 \\ -1 \end{pmatrix}} k$$

An element e of R:

$$e\colon\ 3 \longrightarrow \begin{pmatrix} 5 & 7 \end{pmatrix} \longrightarrow 4$$

Multiplying e with elements of kQ:

$$e \cdot v_1\colon\ 3 \longrightarrow \begin{pmatrix} 0 & 0 \end{pmatrix} \longrightarrow 0$$

$$e \cdot a\colon\ 0 \longrightarrow \begin{pmatrix} 6 & 0 \end{pmatrix} \longrightarrow 0$$

$$e \cdot b\colon\ 0 \longrightarrow \begin{pmatrix} 0 & 0 \end{pmatrix} \longrightarrow 13$$

Representations and relations

Given quiver with relations:

$$Q\colon \begin{tikzcd} 1 \arrow[r, "a"] \arrow[d, "c"'] & 2 \arrow[d, "b"] \\ 3 \arrow[r, "d"] & 4 \end{tikzcd} \qquad \rho = \{ab - 2cd\}$$

A representation M of Q *respects the relation* $ab - 2cd$ if $f_a f_b - 2 f_c f_d = 0$.

$$M\colon \begin{tikzcd} V_1 \arrow[r, "f_a"] \arrow[d, "f_c"'] & V_2 \arrow[d, "f_b"] \\ V_3 \arrow[r, "f_d"] & V_4 \end{tikzcd}$$

Representations and relations

Representations respecting relation $ab - 2cd$?

$$\begin{tikzcd} k \arrow[r, "f_a=1"] \arrow[d, "f_c=1"'] & k \arrow[d, "f_b=\begin{pmatrix}1 & 0\end{pmatrix}"] \\ k \arrow[r, "f_d=\begin{pmatrix}0 & 1\end{pmatrix}"] & k^2 \end{tikzcd}$$

Representations and relations

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NO $(f_a f_b - 2 f_c f_d = \begin{pmatrix}1 & -1\end{pmatrix} \neq 0)$

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NO $(f_af_b - 2f_cf_d = \begin{pmatrix} 1 & -1 \end{pmatrix} \neq 0)$

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$$\begin{tikzcd} k \arrow[r, "f_a=1"] \arrow[d, "f_c=1"'] & k \arrow[d, "f_b=(\,1\;\;0\,)"] \\ k \arrow[r, "f_d=(\,0\;\;1\,)"] & k^2 \end{tikzcd}$$

NO $(f_af_b - 2f_cf_d = \begin{pmatrix} 1 & -1 \end{pmatrix} \neq 0)$

$$\begin{tikzcd} k \arrow[r, "f_a=1"] \arrow[d, "f_c=1"'] & k \arrow[d, "f_b=2"] \\ k \arrow[r, "f_d=1"] & k \end{tikzcd}$$

YES $(f_af_b - 2f_cf_d = 0)$

Representations and relations

For $A = kQ/\langle\rho\rangle$:

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representations of Q over k respecting ρ

Module homomorphisms

For two modules M and N given as representations:

$$M: \qquad V_1 \xrightarrow{f_a} V_2 \xrightarrow{f_b} V_3$$

$$N: \qquad W_1 \xrightarrow{g_a} W_2 \xrightarrow{g_b} W_3$$

Module homomorphisms

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... a homomorphism $h\colon M \to N$ is given by:

Module homomorphisms

For two modules M and N given as representations:

$$\begin{tikzcd} M: \arrow[d, "h"] & V_1 \arrow[r, "f_a"] \arrow[d, "h_1"'] & V_2 \arrow[r, "f_b"] \arrow[d, "h_2"'] & V_3 \arrow[d, "h_3"'] \\ N: & W_1 \arrow[r, "g_a"] & W_2 \arrow[r, "g_b"] & W_3 \end{tikzcd}$$

... a homomorphism $h\colon M \to N$ is given by:

- linear maps h_i for every vertex i,

Module homomorphisms

For two modules M and N given as representations:

$$\begin{tikzcd} M: \arrow[d, "h"] & V_1 \arrow[r, "f_a"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\circ"] & V_2 \arrow[r, "f_b"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\circ"] & V_3 \arrow[d, "h_3"'] \\ N: & W_1 \arrow[r, "g_a"] & W_2 \arrow[r, "g_b"] & W_3 \end{tikzcd}$$

... a homomorphism $h\colon M \to N$ is given by:

- linear maps h_i for every vertex i,
- commuting with the linear maps for the arrows.

Module homomorphisms in QPA

$$M: \qquad 0 \longrightarrow k \xrightarrow{5} k$$

$$N: \qquad k \xrightarrow[\begin{pmatrix} 0 & 3 \end{pmatrix}]{} k^2 \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} k$$

```
gap> Q := Quiver(3, [[1,2,"a"], [2,3,"b"]]);;
gap> kQ := PathAlgebra(Rationals, Q);;
gap> M := RightModuleOverPathAlgebra
          (kQ, [0,1,1], [["b", [[5]]]]);;
gap> N := RightModuleOverPathAlgebra
          (kQ, [1,2,1], [["a", [[0,3]]],
                         ["b", [[1],[1]]]]);;
```

Module homomorphisms in QPA

$$\begin{tikzcd}
M: \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & k \arrow[r, "5"] \arrow[d, "\begin{pmatrix} 3 & 2 \end{pmatrix}"'] & k \arrow[d, "1"] \\
N: & k \arrow[r, "\begin{pmatrix} 0 & 3 \end{pmatrix}"'] & k^2 \arrow[r, "\begin{pmatrix} 1 \\ 1 \end{pmatrix}"] & k
\end{tikzcd}$$

```
gap> h := RightModuleHomOverAlgebra
          (M, N, [ [[0]], [[3,2]], [[1]] ]);
<<[ 0, 1, 1 ]> ---> <[ 1, 2, 1 ]>>
```